Lecture: Conditional Expectation

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Discounted Cash Flow, Section 1.2

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Outline

Uncertainty is a distinguished feature of valuation usually modelled as different future **states of nature** ω with corresponding cash flows $\widetilde{FCF}_t(\omega)$.

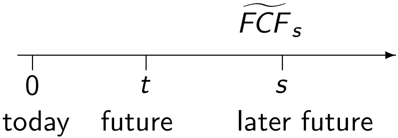
But: to the best of our knowledge particular states of nature play no role in the valuation equations of firms, instead one **uses expectations** $\mathrm{E}\left[\widetilde{FCF}_t\right]$ of cash flows.

Fortune–teller

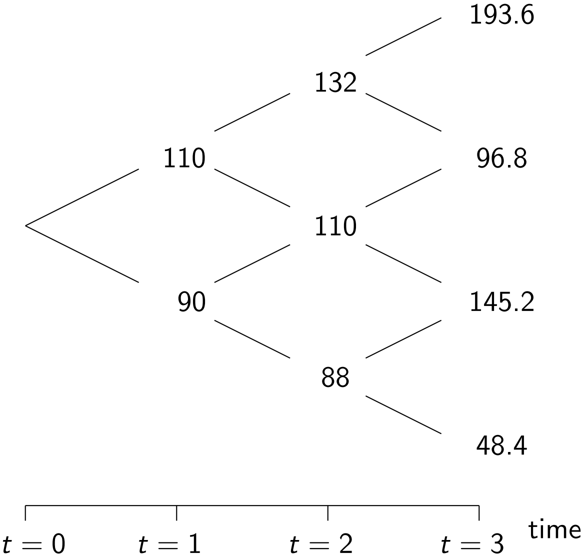
Today is certain, the future is uncertain.

Now: **we always stay at time 0!**

And think about the future



A finite example 3



There are three points in time in the future.

Different cash flow realizations can be observed.

The movements **up** and **down** along the path occur with probability 0.5.

Actual and possible cash flow

Nostradamus (1503–1566),

failed fortune-teller

What happens if actual cash flow at time $t = 1$ is neither 90 nor 110 (for example, 100)?

Our model proved to be wrong!

A.N. Kolmogorov (1903–1987),

founded theory of

conditional expectation

Let us think about cash flow paid at time $t = 3$, i.e. $\widetilde{FCF}_3$. What will its **expectation be tomorrow**?

This depends on the state we will have tomorrow. Two cases are possible: $\widetilde{FCF}_1 = 110$ or $\widetilde{FCF}_1 = 90$.

Case 1 ($\widetilde{FCF}_1 = 110$)

$$\implies \text{Expectation of } \widetilde{FCF}_3 = \frac{1}{4}\times 193.6 + \frac{2}{4}\times 96.8 + \frac{1}{4}\times 145.2 = 133.1.$$

Case 2 ($\widetilde{FCF}_1 = 90$)

$$\implies \text{Expectation of } \widetilde{FCF}_3 = \frac{1}{4}\times 96.8 + \frac{2}{4}\times 145.2 + \frac{1}{4}\times 48.4 = 108.9.$$

Hence, expectation of $\widetilde{FCF}_3$ is

$$\mathsf{E}\left[\widetilde{FCF}_3 | \mathcal{F}_1\right] = \begin{cases} 133.1 & \text{if the development at } t = 1 \text{ is up,} \\ 108.9 & \text{if the development at } t = 1 \text{ is down.} \end{cases}$$

The expectation of $\widetilde{FCF}_3$ depends on the state of nature at time $t = 1$. Hence, the expectation is **conditional**: conditional on the information at time $t = 1$ (abbreviated as $|\mathcal{F}_1$).

A conditional expectation covers our ideas about future thoughts.

This conditional expectation **can be uncertain**.

Arithmetic textbook of

Adam Ries (1492–1559)

How to use conditional expectations? We will not present proofs, but only **rules for calculation**.

The first three rules will be well-known from classical expectations, two will be new.

Rule 1: Classical expectation

$$\boxed{\mathsf{E}\left[\widetilde{X}|\mathcal{F}_0\right] = \mathsf{E}\left[\widetilde{X}\right]}$$

At $t = 0$ conditional expectation and classical expectation coincide.

Or: **conditional expectation generalizes classical expectation.**

Rule 2: Linearity

$$\boxed{\mathsf{E}\left[a\widetilde{X} + b\widetilde{Y}|\mathcal{F}_t\right] = a\,\mathsf{E}\left[\widetilde{X}|\mathcal{F}_t\right] + b\,\mathsf{E}\left[\widetilde{Y}|\mathcal{F}_t\right]}$$

Business as usual . . .

Rule 3: Certain quantity

$$\boxed{\mathsf{E}\left[1|\mathcal{F}_t\right] = 1}$$

Safety first...

From this and linearity for certain quantities X,

$$\begin{aligned} \mathsf{E}\left[X|\mathcal{F}_t\right] &= \mathsf{E}\left[X1|\mathcal{F}_t\right] \\ &= X\,\mathsf{E}\left[1|\mathcal{F}_t\right] \\ &= X \end{aligned}$$

Rule 4: Iterated expectation

Let $s \geq t$ then

$$\boxed{\mathsf{E}\left[\mathsf{E}\left[\widetilde{X}|\mathcal{F}_s\right]\ |\mathcal{F}_t\right] = \mathsf{E}\left[\widetilde{X}|\mathcal{F}_t\right]}$$

When we think today about what we will know tomorrow about the day after tomorrow,

we will only know what we today already believe to know tomorrow.

Rule 5: Known or realized quantities

If $\widetilde{X_t}$ is known at time t

$$\boxed{\mathsf{E}\left[\widetilde{X_t}\,\widetilde{Y}|\mathcal{F}_t\right] = \widetilde{X_t}\,\mathsf{E}\left[\widetilde{Y}|\mathcal{F}_t\right]}$$

We can take out from the expectation what is known.

Or: **known quantities are like certain quantities.**

We want to check our rules by looking at the finite example and an infinite example. We start with the finite example:

Remember that we had

$$\mathsf{E}\left[\widetilde{FCF}_3|\mathcal{F}_1\right] = \begin{cases} 133.1 & \text{if up at time } t = 1 \ , \\ 108.9 & \text{if down at time } t = 1 \ . \end{cases}$$

From this we get

$$\mathsf{E}\left[\mathsf{E}\left[\widetilde{FCF}_3|\mathcal{F}_1\right]\right] = \frac{1}{2} \times 133.1 + \frac{1}{2} \times 108.9 = 121.$$

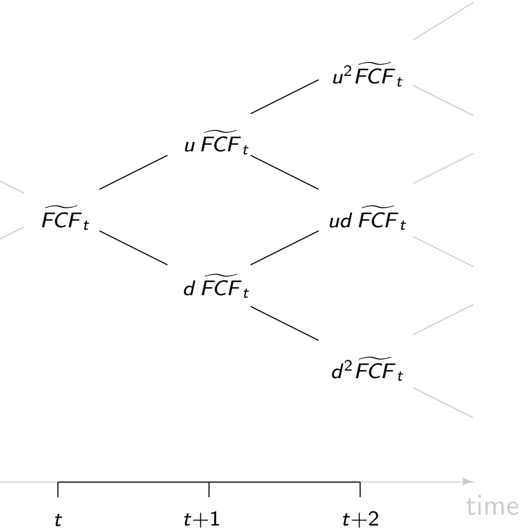
And indeed

$$
\begin{aligned}
\mathsf{E}\left[\mathsf{E}\left[\widetilde{FCF}_3|\mathcal{F}_1\right]\right] &= \mathsf{E}\left[\mathsf{E}\left[\widetilde{FCF}_3|\mathcal{F}_1\right]|\mathcal{F}_0\right] && \text{by rule 1} \\
&= \mathsf{E}\left[\widetilde{FCF}_3|\mathcal{F}_0\right] && \text{by rule 4} \\
&= \mathsf{E}\left[\widetilde{FCF}_3\right] && \text{by rule 1} \\
&= 121\ !
\end{aligned}
$$

It seems **purely by chance** that

$$\mathrm{E}\left[\widetilde{FCF}_3|\mathcal{F}_1\right] = 1.1^{3-1} \times \widetilde{FCF}_1\,,$$

but it is on purpose! This will become clear later (when discussing autoregressive cash flows).



Again two factors up and down with probability p_u and p_d and $0 < d < u$

or

$$\widetilde{FCF}_{t+1} = \begin{cases} u\widetilde{FCF}_t & \text{up,} \\ d\widetilde{FCF}_t & \text{down.} \end{cases}$$

Let us evaluate the conditional expectation

$$\begin{aligned} \mathsf{E}\left[\widetilde{FCF}_{t+1}|\mathcal{F}_t\right] &= p_u u \widetilde{FCF}_t + p_d d \widetilde{FCF}_t \\ &= \underbrace{(p_u u + p_d d)}_{:=1+g} \widetilde{FCF}_t \,, \end{aligned}$$

where g is the expected growth rate.

If $g = 0$ it is said that the cash flows 'form a martingal'. In the infinite example we will later assume no growth ($g = 0$).

This can be extended if $s > t$

$$
\begin{aligned}
\mathsf{E}\left[\widetilde{FCF}_s|\mathcal{F}_t\right] &= \mathsf{E}\left[\mathsf{E}\left[\widetilde{FCF}_s|\mathcal{F}_{s-1}\right]|\mathcal{F}_t\right] && \text{by rule 4}\\
&= \mathsf{E}\left[(1+g)\widetilde{FCF}_{s-1}|\mathcal{F}_t\right] && \text{see above}\\
&= (1+g)\,\mathsf{E}\left[\widetilde{FCF}_{s-1}|\mathcal{F}_t\right] && \text{by rule 2}\\
&= (1+g)^{s-t}\,\mathsf{E}\left[\widetilde{FCF}_t|\mathcal{F}_t\right] && \text{repeating argument}\\
&= (1+g)^{s-t}\widetilde{FCF}_t && \text{by rule 5 and rule 3}
\end{aligned}
$$

Summary

We always stay in the present. Conditional expectation handles our knowledge of the future.

Five rules cover the necessary mathematics.